Revisiting Enumerative Instantiation

Andrew Reynolds[1], Haniel Barbosa[1,2] and Pascal Fontaine[1]

[1]University of Iowa, Iowa City, U.S.A.

[2]University of Lorraine, CNRS, Inria, LORIA, Nancy, France

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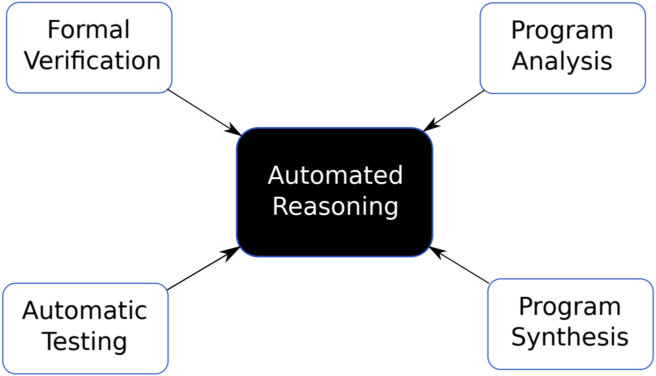
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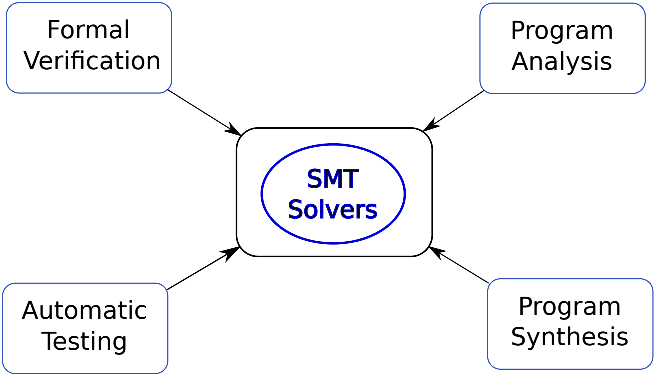
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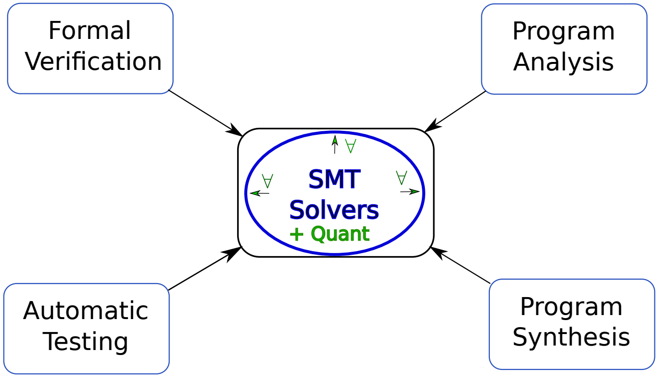
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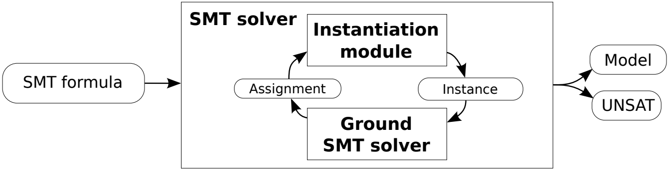
Outline

- ▷ Quantifier handling in SMT solving
- ▷ Strengthening the Herbrand Theorem
- ▷ Effective enumerative instantiation
  - ▶ Combination with other instantiation strategies
  - ▶ Implementation
- ▷ Evaluation

Quantifier handling in SMT

Problem statement



Quantifier-free solver enumerates assignments $\mathsf{E} \cup \mathsf{Q}$

- E is a set of ground literals $\{a \leq b,\, b \leq a + x,\, x \simeq 0,\, f(a) \not\simeq f(b)\}$
- Q is a set of quantified clauses $\{\forall xyz.\, f(x) \not\simeq f(z) \vee g(y) \simeq h(z)\}$

Instantiation module generates instances of Q $f(a) \not\simeq f(b) \vee g(a) \simeq h(b)$

Instantiation strategies: *trigger-based* [Detlefs et al. J. ACM'05]

Trigger-based instantiation (E-matching): search for relevant instantiations according to a set of triggers and E-matching

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Trigger-based instantiation (E-matching): search for relevant instantiations according to a set of triggers and E-matching

- ▷ $\mathsf{E} = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $\mathsf{Q} = \{\forall x.\, P(x) \vee R(x)\}$
- ▷ Assume the set of triggers $\{(P(x))\}$.
- ▷ Since $\mathsf{E} \models P(x)\{x \mapsto t\} \simeq P(t)$, for $t = a, b, c$, this strategy may return $\{\{x \mapsto a\}, \{x \mapsto b\}, \{x \mapsto c\}\}$.
- ▷ Formally:

$\mathbf{e}(\mathsf{E}, \forall \bar{x}.\, \varphi)$:

1. Select a set of triggers $\{\bar{t}_1, \dots \bar{t}_n\}$ for $\forall \bar{x}.\, \varphi$.
2. For each $i = 1, \dots, n$, select a set of substitutions S_i s.t. for each $\sigma \in S_i$, $\mathsf{E} \models \bar{t}_i\sigma \simeq \bar{g}_i$ for some tuple $\bar{g}_i \in \mathbf{T}(\mathsf{E})$.
3. Return $\bigcup_{i=1}^{n} S_i$.

Instantiation strategies: *conflict-based* [Reynolds et al. FMCAD'14]

Conflict-based instantiation: search for instantiations of a quantified formula in Q that make E unsatisfiable

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- $\mathsf{E} = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $\mathsf{Q} = \{\forall x.\, P(x) \vee R(x)\}$
- Since E, $P(b) \vee R(b) \models \bot$, this strategy will return $\{\{x \mapsto b\}\}$.
- Formally:

  $\mathbf{c}(\mathsf{E},\, \forall \bar{x}.\, \varphi)$: 1. Either return $\{\sigma\}$ where $\mathsf{E}, \varphi\sigma \models \bot$, or return $\emptyset$.

Instantiation strategies: *model-based* [Ge and de Moura CAV'09]

Model-based instantiation (MBQI): build a candidate model for $E \cup Q$ and instantiate with counter-examples from model checking

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Model-based instantiation (MBQI): build a candidate model for $\mathsf{E} \cup \mathsf{Q}$ and instantiate with counter-examples from model checking

- $\mathsf{E} = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $\mathsf{Q} = \{\forall x.\, P(x) \vee R(x)\}$
- Assume that $P^{\mathcal{M}} = \lambda x.\, \mathsf{ite}(x \simeq c,\, \top,\, \bot)$ and $R^{\mathcal{M}} = \lambda x.\, \bot$.
- Since $\mathcal{M} \not\models P(a) \vee R(a)$, this strategy may return $\{\{x \mapsto a\}\}$.
- Formally:

$\mathbf{m}(\mathsf{E}, \forall \bar{x}.\, \varphi)$:
1. Construct a model $\mathcal{M}$ for E.
2. Return $\{\{\bar{x} \mapsto \bar{t}\}\}$ where $\bar{t} \in \mathbf{T}(\mathsf{E})$ and $\mathcal{M} \not\models \varphi\{\bar{x} \mapsto \bar{t}\}$, or $\emptyset$ if none exists.

Shortcomings

- ▷ Conflict-based instantiation (**c**):
  - Inherently incomplete
- ▷ E-matching (**e**):
  - Too many instances
  - Butterfly effect
- ▷ MBQI (**m**):
  - Complete for many fragments, but slow convergence for UNSAT
  - Better suited for model finding

Generally SMT solvers implement complete techniques by applying **m** as a last resort after trying **c** and **e**

Strengthening the Herbrand Theorem

Why can we use instantiation?

Theorem (Herbrand)

A set of pure first-order logic formulas is unsatisfiable if and only if there exists a finite unsatisfiable set of its instances.

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- The earliest theorem provers relied on *Herbrand instantiation*
  - Instantiate with all possible terms in the language
- Enumerating all instances is unfeasible in practice!
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We make enumerative instantiation beneficial for state-of-the-art SMT

- ▷ strengthening of Herbrand theorem
- ▷ efficient implementation techniques

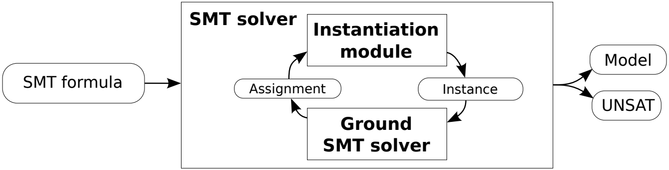
Theorem (Strengthened Herbrand)

If there exists an infinite series of finite satisfiable sets of ground literals E_i *and of finite sets of ground instances* Q_i *of* Q *such that*

- ▷ $\mathsf{Q}_i = \big\{\varphi\sigma \;\mid\; \forall\bar{x}.\,\varphi \in \mathsf{Q},\, \mathsf{dom}(\sigma) = \{\bar{x}\} \wedge \mathsf{ran}(\sigma) \subseteq \mathbf{T}(\mathsf{E}_i)\big\}$;
- ▷ $\mathsf{E}_0 = \mathsf{E}$, $\mathsf{E}_{i+1} \models \mathsf{E}_i \cup \mathsf{Q}_i$;

then $\mathsf{E} \cup \mathsf{Q}$ *is satisfiable in the empty theory with equality.*

Direct application at



- ▷ Quantifier-free solver enumerates assignments $\mathsf{E} \cup \mathsf{Q}$
- ▷ Instantiation module generates instances of Q

Effective enumerative instantiation

Enumerative instantiation

$\mathbf{u}(\mathsf{E}, \forall \bar{x}.\, \varphi)$:

1. Choose an ordering $\preceq$ on tuples of quantifier-free terms.
2. Return $\{\{\bar{x} \mapsto \bar{t}\}\}$ where $\bar{t}$ is a minimal tuple of terms w.r.t $\preceq$, such that $\bar{t} \in \mathbf{T}(\mathsf{E})$ and $\mathsf{E} \not\models \varphi\{\bar{x} \mapsto \bar{t}\,\}$, or $\emptyset$ if none exist.

- $\triangleright$ $\mathsf{E} = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $\mathsf{Q} = \{\forall x.\, P(x) \vee R(x)\}$
- $\triangleright$ $\mathbf{u}$ chooses an ordering on tuples of terms, say the lexicographic extension of $\preceq$ where $a \prec b \prec c$.
- $\triangleright$ Since E does not entail $P(a) \vee R(a)$, this strategy returns $\{\{x \mapsto a\}\}$.

u as an alternative for **m**

- ▷ Enumerative instantiation plays a similar role to MBQI
- ▷ It can also serve as a "completeness fallback" to **c** and **e**
- ▷ However, **u** has advantages over **m** for UNSAT problems
- ▷ Moreover it is significantly simpler to implement
  - ▶ No model building
  - ▶ No model checking

Example

$$\begin{array}{lcl} \mathsf{E} & = & \{\neg P(a),\ R(b),\ S(c)\} \\ \mathsf{Q} & = & \{\forall x.\, R(x) \vee S(x),\ \forall x.\, \neg R(x) \vee P(x),\ \forall x.\, \neg S(x) \vee P(x)\} \end{array}$$

$$\mathsf{M} = \left\{ \begin{array}{lcl} P^{\mathcal{M}} & = & \lambda x.\, \bot, \\ R^{\mathcal{M}} & = & \lambda x.\, \mathsf{ite}(x \simeq b,\, \top,\, \bot), \\ S^{\mathcal{M}} & = & \lambda x.\, \mathsf{ite}(x \simeq c,\, \top,\, \bot) \end{array} \right\}, \qquad a \prec b \prec c$$

φ	x s.t. $\mathcal{M} \not\models \varphi$	x s.t. $\mathsf{E} \not\models \varphi$	$\mathbf{m}(\mathsf{E}, \forall x.\, \varphi)$	$\mathbf{u}(\mathsf{E}, \forall x.\, \varphi)$
$R(x) \vee S(x)$	a	a	$\{\{x \mapsto a\}\}$	$\{\{x \mapsto a\}\}$
$\neg R(x) \vee P(x)$	b	a, b, c	$\{\{x \mapsto b\}\}$	$\{\{x \mapsto a\}\}$
$\neg S(x) \vee P(x)$	c	a, b, c	$\{\{x \mapsto c\}\}$	$\{\{x \mapsto a\}\}$

- $\triangleright$ **u** instantiates uniformly so that new terms are introduced less often
- $\triangleright$ **m** instantiates depending on how model was built
- $\triangleright$ Moreover, **u** leads to $\mathsf{E} \wedge \mathsf{Q}\{x \mapsto a\} \models \bot$
- $\triangleright$ **m** requires considering E' which satisfies E along the new instances

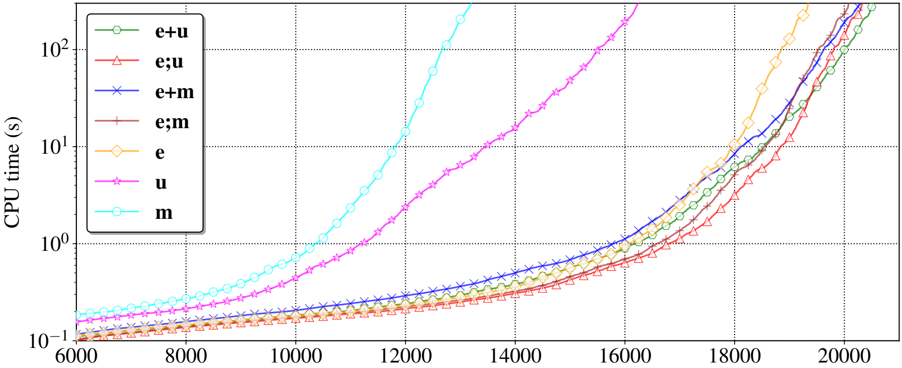
Implementation

Implementing enumerative instantiation efficiently depends on:

- ▷ Restricting enumeration space
- ▷ Avoiding entailed instantiations
- ▷ Term ordering to introduce new terms less often

Evaluation

CVC4 configurations on unsatisfiable benchmarks



- $42\,065$ benchmarks, being $14\,731$ from TPTP and $27\,334$ from SMT-LIB
- **e**+**u** stands for "interleave **e** and **u**", while **e**;**u** for "apply **e** first, then **u** if it fails"
- All CVC4 configurations have "**c**;" as prefix

Impact of **u** on satisfiable benchmarks

Library	#	**u**	**e;u**	**e+u**	**e**	**m**	**e;m**	**e+m**
TPTP	14731	471	492	464	17	930	808	829
UF	7293	39	42	42	0	70	69	65
Theories	20041	3	3	3	3	350	267	267
Total	42065	513	537	509	20	1350	1144	1161

- ▷ As expected, **m** greatly outperforms **u**
- ▷ Nevertheless **u** answers SAT *half as often* as **m** in empty theory
- ▷ Moreover, **u** solves 13 problems **m** does not

Conclusions

- We have introduced an efficient way of applying enumerative instantiation in SMT solving
- New technique is based on an strengthening of the Herbrand Theorem
- Implementation in SMT solver CVC4
  - Significantly increases success rate
  - Outperforms existing implementations of MBQI for UNSAT
  - Can be used for SAT in the empty theory

Appendix

Restricting Enumeration Space

- Strengthened Herbrand Theorem allows restriction to $\mathbf{T}(\mathsf{E})$
- *Sort inference* reduces instantiation space by computing more precise sort information
  - $\mathsf{E} \cup \mathsf{Q} = \{a \not\simeq b,\ f(a) \simeq c\} \cup \{P(f(x))\}$
    - $a, b, c, x : \tau$
    - $f : \tau \to \tau$ and $P : \tau \to$ Bool.
  - This is equivalent to
    $\mathsf{E}^s \cup \mathsf{Q}^s = \{a_1 \not\simeq b_1,\ f_{12}(a_1) \simeq c_2\} \cup \{P_2(f_{12}(x_1))\}$
    - $a_1, b_1, x_1 : \tau_1$
    - $c_2 : \tau_2$
    - $f_{12} : \tau_1 \to \tau_2$ and $P : \tau_2 \to$ Bool
  - **u** would derive e.g. $\{x \mapsto c\}$ for $\mathsf{E} \cup \mathsf{Q}$, while for $\mathsf{E}^s \cup \mathsf{Q}^s$ the instantiation $\{x_1 \mapsto c_2\}$ is not well-sorted.

Entailment Checks

Two-layered method for checking whether E $\models \varphi\{\bar{x} \mapsto \bar{t}\}$ holds

- ▷ Cache of instantiations already derived
- ▷ Incomplete but fast method for checking E $\models \ell$

  Repeat until a fix point:

  1. Replace each leaf term t in ℓ with $[t]$.
  2. Replace each term $f(t_1, \ldots, t_n)$ in ℓ with s if $(t_1, \ldots, t_n) \to s \in \mathcal{I}_f$.
  3. Replace each term $f(t_1, \ldots, t_n)$ in ℓ where f is an interpreted function with the result of the evaluation $f(t_1, \ldots, t_n)\!\downarrow$.

  Then, if the resultant ℓ is $\top$, then the entailment holds.

  - ▶ Extension to incorporate Boolean structure
  - ▶ Extension to other theories through theory-specific rewriting

Term Ordering

Instantiations are enumerated according to the order

$$(t_1, \dots, t_n) \prec (s_1, \dots, s_n) \quad \text{if} \quad \begin{cases} \max_{i=1}^{n} t_i \prec \max_{i=1}^{n} s_i, \text{ or} \\ \max_{i=1}^{n} t_i = \max_{i=1}^{n} s_i \text{ and} \\ \qquad\qquad (t_1, \dots, t_n) \prec_{\mathsf{lex}} (s_1, \dots, s_n) \end{cases}$$

for a given order $\preceq$ on ground terms.

If $a \prec b \prec c$, then

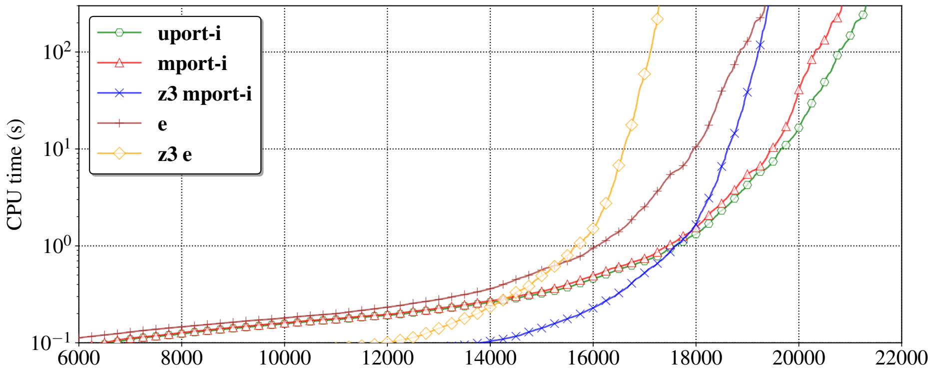
$$(a, a) \prec (a, b) \prec (b, a) \prec (b, b) \prec (a, c) \prec (c, b) \prec (c, c)$$

- We consider instantiations with c only after considering all cases with a and b
- Goal is to introduce new terms less often
- Order on $\mathbf{T}(\mathsf{E})$ fixed for finite set of terms $t_1 \prec \dots \prec t_n$
  - Instantiate in order with $t_1, \dots, t_n$
  - Then choose new non-congruent term $t \in \mathbf{T}(\mathsf{E})$ and have $t_n \prec t$

Impact of **u** on unsatisfiable benchmarks

- **u** solves $3\,043$ more benchmarks than **m**
- **u** solves $1\,737$ problems not solvable by **e**
- Combinations of **e** with **u** or **m** lead to significant gains
- **e**+**u** is best configuration, solving 253 more problems than **e**+**m** and $1\,295$ more than **e**
- Some benchmark families only solvable due to enumeration
- Overall the enumerative strategies lead to a virtual portfolio of CVC4 solving 712 more problems

Comparison against other instantiation-based SMT solvers



- ▷ Portfolios run without interleaving strategies (not supported by Z3)
- ▷ Z3 uses several optimizations for **e** not implemented in CVC4
- ▷ Z3 does not implement **c**